

## MMAT 5010 Linear Analysis (2022-23): Homework 1.

Deadline: 24 Sep 2022

### Important Notice:

- ♣ The answer paper must be submitted before the deadline.
- ♠ The answer paper MUST BE sent to the CU Blackboard. Please refer to the course web for details.

1. Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be the normed spaces over the field  $\mathbb{K}$ . Let  $T : X \rightarrow Y$  be a linear map from  $X$  to a  $Y$ , that is  $T$  satisfies the condition:  $T(sx + tx') = sTx + tTx'$  for all  $x, x'$  in  $X$  and  $s, t \in \mathbb{K}$ . For each  $x \in X$ , let  $q(x) := \|x\|_X + \|Tx\|_Y$ . Show that  $q$  is a norm function on  $X$ .
2. Let  $X$  be a normed space. Show that  $X$  is a Banach space if and only if the unit sphere  $S := \{x \in X : \|x\| = 1\}$  of  $X$  is complete, that is, every Cauchy sequence  $(x_n)$  in  $S$  has the limit in  $S$ .
3. Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be the normed spaces over the field  $\mathbb{K}$ . The direct sum of  $X$  and  $Y$ , write  $X \oplus Y$ , is defined by  $X \oplus Y := \{(x, y) : x \in X, y \in Y\}$  under the addition and the scalar multiplication:  $(x, y) + (x', y') := (x + x', y + y')$  and  $t(x, y) := (tx, ty)$  for  $(x, y); (x', y') \in X \oplus Y$  and  $t \in \mathbb{K}$ .  
For each  $(x, y) \in X \oplus Y$ , let

$$q(x, y) := \|x\|_X + \|y\|_Y.$$

- (a) Show that  $q$  is a norm function on  $X \oplus Y$ .
- (b) Show that  $X \oplus Y$  is a Banach space under the norm  $q$  if and only if  $X$  and  $Y$  both are Banach spaces.

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